

Problem set 5

Problem and exercise numbers are from CLRS 2nd edition.

1. Let $G = (V, E, w)$ be a complete positively-weighted ($w > 0$) graph with even number of vertices, and let M be a perfect matching in it. Give an algorithm to test whether M is a minimum-weight perfect matching. (Hint: Negate the weights of edges in M and consider the symmetric difference between M and the minimum-weight perfect matching.)
2. Prove that any tree is a bipartite graph.
3. Let $G = (A \cup B, E)$ be a bipartite graph with $|A| = |B|$. True or False: G has a perfect matching.
4. Give an algorithm to find a shortest Chinese postman *path*, i.e., a tour that starts at a given vertex s , ends at a given vertex $t \neq s$ and traverses each edge of the graph.
5. Give a Christofides-like algorithm to find an approximate TSP *path*, i.e., a tour that starts at a given vertex s , ends at a given vertex $t \neq s$ and visits each vertex of the graph.
6. Let $G = (V, E, w)$ be a complete graph, where w is a metric; that is, G is undirected ($w_{ij} = w_{ji} \forall i, j \in V$), $w \geq 0$ and $w_{ij} \leq w_{ik} + w_{kj} \forall i, j, k \in V$. Let $CP(G)$ be the length of the optimal Chinese postman tour, and let $Cr(G)$ be the length of the tour produced by the Christofides algorithm. Show that $Cr(G) \leq CP(G)$.
7. Let $G = (V, E, w)$ be a complete graph, where w is a metric; that is, G is undirected ($w_{ij} = w_{ji} \forall i, j \in V$), $w \geq 0$ and $w_{ij} \leq w_{ik} + w_{kj} \forall i, j, k \in V$. Let $TSP(G)$ be the length of the optimal TSP tour, and let $Cr(G)$ be the length of the tour produced by the Christofides algorithm; let

$$\rho(G) = \frac{Cr(G)}{TSP(G)}$$

be the approximation ratio of the Christofides algorithm run on G .

We know that $\forall G, \rho(G) \leq \frac{3}{2}$. How close to $\frac{3}{2}$ can $\rho(G)$ be? Give a generic example of graph G for which $\rho(G)$ is arbitrarily close to $\frac{3}{2}$. More formally, construct an infinite sequence of graphs (G_1, G_2, \dots) such that G_n has n vertices and

$$\limsup_{n \rightarrow \infty} \rho(G_n) = \frac{3}{2}.$$

That is, when n goes to infinity, the approximation ratio of the Christofides algorithm on G_n should tend to $\frac{3}{2}$.

Extra credit: Suppose the weight of every edge G is either 1 or 2; i.e., $w : E \mapsto \{1, 2\}$. What is the largest that $\rho(G)$ can be in this case? Can it be larger than $\frac{8}{7}$? Can it be larger than $\frac{7}{5}$?