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Abstract— **In simulation studies the focus has traditionally been on time-independent characteristics of queueing systems, particularly on the steady-state mean. The systems, even if they are in stochastic equilibrium, have also time-dependent or dynamic properties which are of great practical importance. Typically, the dynamic properties are related to the variation of traffic or load during some period of time.**

In this paper we propose a way to examine dynamic properties of queueing systems through simulation. Our approach is based on the following idea: The original traffic rates are classified into low and heavy values. The resulting binary sequence of indicators characterize the traffic level of the system. The variation in the traffic level is summarized through estimating the two histograms of level durations. The rationale of translating the problem into simultaneous estimation of two histograms is that the estimates for the cell proportions are based on cumulative statistics. Therefore, we can use the Bonferroni inequality and the existing methods for assessing the precisions of estimated proportions.

Keywords— **dynamic properties of queueing models, simulation estimation of proportions, simultaneous assessment of precisions.**

I. Introduction

In simulation studies the focus has traditionally been on time-independent characteristics of queueing systems – such as the mean, variance, percentiles, and proportions. The systems, even if they are in stochastic equilibrium, have also time-dependent or dynamic properties which are of great practical importance. Typically, the dynamic properties are related to the variation of traffic or load during some period of time. In this paper we propose a way to examine dynamic properties of queueing systems through simulation.

When dynamic properties are examined, the fundamental problem is the characterization: Which set of indices properly summarizes the dynamic properties? The second problem is, how we can assess the precision of estimated indices. As noted in [1], a simulation study of any stochastic process is only a programming exercise if the output analysis of the results is not properly carried out; see also [2].

We propose one possible way to characterize and estimate dynamic properties of queueing systems. The variety of dynamic properties that different queueing systems may have is so huge that a uniform approach is impossible. Yet, we believe that the approach we propose can be used in many situations. The approach is based on the following idea: *The original traffic rates $\{X_i\}_{i=1}^N$ are classified into low and heavy values. The resulting binary sequence of indicators $\{I_i\}_{i=1}^N$ characterize the traffic levels of the system. The variation of traffic levels is summarized through estimating the two histograms of traffic level durations.*

The rationale of translating the problem into simultaneous estimation of two histograms is that the estimates for the cell proportions are based on cumulative statistics. Therefore, we can use the Bonferroni inequality and the existing methods for assessing the precisions of estimated proportions.

The most widely used method for the assessment of precision is to estimate a confidence interval. The methods for confidence interval estimation of the steady-state mean are well-known. Available methods are described in many textbooks and survey articles including [1; 3; 4; 5]. As explained in [5], these methods can be used to estimate a confidence interval for other indices if the indices can be estimated through cumulative statistics. A method for controlling the precision of several simultaneous means can be found in [6]. The simulation estimation of proportions is studied in [7].

In Section II we introduce the transforms and indices that we use to characterize the dynamic properties of the original output sequence. In Section III we discuss the method for estimating the proportions. In Section IV we report empirical results.

II. An Approach to Summarize Dynamic Properties

The variety of dynamic properties that queueing systems may pose is very rich. This diversity makes it difficult or even impossible to discuss those properties in general. Therefore, we concentrate on properties which are important and which are easy enough to analyze. By easiness we mean that methods for assessing the precision of the estimates are available. The easiness implies that the estimates are based on cumulative statistics.

As mentioned in the Introduction, the dynamic properties are typically related to the changes in traffic rate during some periods of time. In order to make the subject concrete, we give an example. Figure 1 shows the measured packet rate to and from a file server.

The traffic rate being examined usually has one natural threshold value. Typically, the threshold results in traffic levels of *low* and *heavy traffic*. In this case, the analyst may be interested in how long a time the traffic rate remains be-

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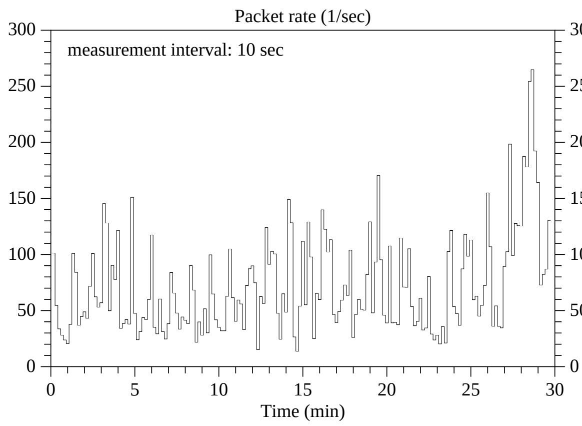


Fig. 1. Measured Packet Rate

low and above the given threshold. It may be sufficient to learn the means of the two durations. However, in many cases the analyst wants to learn the distributions, typically in the form of histogram. Therefore, we estimate the histograms of durations that the traffic rate remains below and above the given threshold value.

In addition to the threshold value ($x_{threshold}$), the analyst need to specify the class boundaries in the histograms being estimated. Quite often the application suggests appropriate boundaries. If the analyst believes that the distribution may be (very) long-tailed, boundaries that are powers of 2 are usually a good ad hoc guess. This kind of classification is called *octave bins* in [8].

We need three successive transforms to obtain the sequences of indicators from which the cell proportions of the histograms are estimated. In the first phase the original sequence $\{X_j\}_{j=1}^N$ is transformed into binary sequence $\{I_j\}_{j=1}^N$:

$$I_j = \begin{cases} 0, & \text{if} \quad X_j \le x_{threshold} \, , \\ 1, & \text{if} \quad X_j > x_{threshold} \, . \end{cases}$$

In the second phase the binary sequence is transformed into twodimensional sequence of durations: $\{Y_{0,k}, Y_{1,k}\}_{k=1}^M$, where $Y_{0,k}$ is the number of consecutive zeros in the kth cycle and $Y_{1,k}$ is the number of consecutive ones in the kth cycle. A cycle consists of one consecutive sequence of zeros and ones:

$$\overbrace{00\underbrace{11110}_{\text{cycle 1}}\underbrace{1100000}_{\text{cycle 2}}1\ldots\ldots\ldots0\underbrace{111110}_{\text{cycle } M}11000}^{\text{sequence length } N}$$

At both ends of the binary sequence we have one incomplete cycle. In the example above, we do not know how many zeros precede the first two observed zeros. Similarly, we do not yet know how many zeros will follow the last three observed zeros.

In the third phase the twodimensional sequence of durations, $\{Y_{0,k}, Y_{1,k}\}_{k=1}^M$, is transformed into $(C_0 + C_1)$-dimensional sequence of indicators:

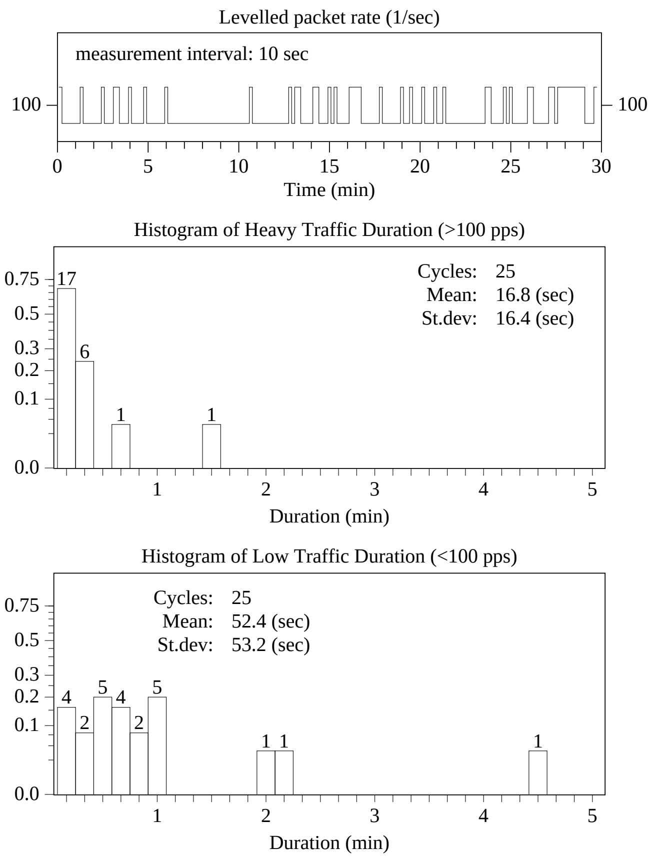


Fig. 2. Summary of Levelled Packet Rate

$$I_{j,k} = \begin{cases} 1, & \text{if } c_{0,j-1} < Y_{0,k} \le c_{0,j} \\ 0, & \text{otherwise} \end{cases} \, , \tag{1}$$

$j = 1, \ldots, C_0$, and

$$I_{C_0+j,k} = \begin{cases} 1, & \text{if } c_{1,j-1} < Y_{1,k} \le c_{1,j} \\ 0, & \text{otherwise} \end{cases} \, , \tag{2}$$

$j = 1, \ldots, C_1$. In (1) the $c_{0,j}$'s are the class boundaries for the low traffic duration and in (2) the $c_{1,j}$'s for the high traffic duration.

Figure 2 displays the levelled packet rate and the histograms of high and low traffic durations.

III. Simultaneous Estimation of Several Proportions

In this section we describe the main points of the method for controlling the precision the estimated proportions. We start by discussing briefly the estimation of the proportions from the indicator sequences. Secondly, we outline the derivation of approximate confidence intervals for the proportions. Our third topic is the precision requirement and the termination rule. The rule proposed can not be used if a proportion is close to 0 or 1. Therefore, we complete this section by discussing the treatment of small proportions (rare events). The detailed description of the method can be found in [7].

A. Estimating Several Proportions Simultaneously

When several proportions are estimated, we are interested in the probabilities $p_1, \ldots, p_m$ that the response variable X belongs to prespecified categories $\mathcal{C}_1, \ldots, \mathcal{C}_m$: $p_k = \Pr\{X \in \mathcal{C}_k\}$, $k = 1, \ldots, m$. Each estimate is based on a transformation of the output sequence $\{X_j\}_{j=1}^n$:

$$\hat{p}_k = \frac{1}{n}\sum_{j=1}^{n} I_{k,j} \ ,$$

where $I_{k,j} = 1$ if $X_j \in \mathcal{C}_k$, and $I_{k,j} = 0$ if $X_j \notin \mathcal{C}_k$. The categories are not necessarily disjoint. In addition, they need not cover all possible values of the response variable. Therefore, $\sum_{k=1}^m p_k$ may be less than 1, equal to 1, or greater than 1.

B. Confidence Intervals for the Proportions

When the indicator sequences $\{I_{k,j}\}$ are covariance stationary, then the $\hat{p}_k$'s have normal limiting distributions when $0 < p_k < 1$. Although each $\hat{p}_k$ has a normal limiting distribution, the confidence intervals based on the normal approximation are not always appropriate. One of the methods proposed to improve the approximate confidence interval for p_k, $[\hat{p}_{L_k}, \hat{p}_{U_k}]$, is based on the arcsin-transform ($\hat{y}_k = 2\arcsin\sqrt{\hat{p}_k}$; see e.g. [9, p. 699]), and on approximating the distribution of $(\hat{y}_k - y_k)/s_{\hat{y}_k}$ ($s^2_{\hat{y}_k}$ is the estimated variance of $\hat{y}_k$ with ν_k degrees of freedom) by the Student t-distribution with ν_k degrees of freedom.

When $\hat{y}_k = 2\arcsin\sqrt{\hat{p}_k}$, then $\mathrm{E}(\hat{y}_k) \approx 2\arcsin\sqrt{p_k}$ and $\mathrm{Var}(\hat{y}_k) \approx \mathrm{Var}(\hat{p}_k)/(p_k - p_k^2)$ when n is large and $0 < p_k < 1$. This is obtained using the Taylor series representation; see e.g. [10, pp. 261 and 275]. Therefore, a natural estimate of $\mathrm{Var}(\hat{y}_k)$ is $s^2_{\hat{p}_k}/(\hat{p}_k - \hat{p}_k^2)$. Now we obtain the following approximate confidence intervals, $[\hat{p}_{L_k}, \hat{p}_{U_k}]$, for the proportions p_k , $k = 1, \ldots, m$, at confidence levels $1 - \alpha_k$:

$$\begin{aligned} \hat{p}_{L_k} &= \sin^2(\max(0, \hat{y}_{L_k}/2)) \ , \\ \hat{p}_{U_k} &= \sin^2(\min(\pi/2, \hat{y}_{U_k}/2)) \ , \end{aligned}$$

where

$$\begin{aligned} \hat{y}_{L_k} &= 2\arcsin\sqrt{\hat{p}_k} - \\ &\quad t_{\nu_k}(1-\alpha_k/2)s_{\hat{p}_k}/\sqrt{\hat{p}_k(1-\hat{p}_k)} \ , \\ \hat{y}_{U_k} &= 2\arcsin\sqrt{\hat{p}_k} + \\ &\quad t_{\nu_k}(1-\alpha_k/2)s_{\hat{p}_k}/\sqrt{\hat{p}_k(1-\hat{p}_k)} \ , \end{aligned}$$

and the $s^2_{\hat{p}_k}$'s are the estimated variances of the $\hat{p}_k$'s, each with ν_k degrees of freedom, and $t_{\nu_k}(x)$ is the $100x$th percentile of the Student t-distribution with ν_k degrees of freedom.

Since the $\hat{p}_k$'s are based on cumulative statistics, the methods developed for estimating the variance of the mean can be used to estimate $s^2_{\hat{p}_k}$. We use the spectral method introduced in [11] to estimate the variances. The method is widely recognized as one of the most robust and reliable method for estimating the variance of a steady-state mean; see e.g. [1].

C. Controlling the Precision of Estimated Proportions

The relative half-width criterion is commonly used to specify the target precision of an estimated mean. The analyst specifies the desired relative half-width ε of the $100(1-\alpha)\%$ confidence interval for the unknown mean θ. Hence, the desired confidence interval for θ is $[\hat{\theta}(1-\varepsilon), \hat{\theta}(1+\varepsilon)]$ where $\hat{\theta}$ is the estimated mean. The methods controlling the precision terminate the simulation as soon as the desired confidence interval includes the estimated confidence interval, i.e. $(1-\varepsilon)\hat{\theta} \le \hat{\theta}_L < \hat{\theta}_U \le (1+\varepsilon)\hat{\theta}$; $[\hat{\theta}_L, \hat{\theta}_U]$ is the estimated $100(1-\alpha)\%$ confidence interval.

The primary reason for using relative half-width (instead of the absolute one) is that the analyst is interested in the magnitude and in one or two leading digits of the mean. However, the relative half-width is not appropriate for proportions. Typically, the closer to the end points of the interval $[0, 1]$ the proportion is, the narrower the confidence interval requested is. There are several ways of selecting the desired width of the confidence interval. We use the arcsin transform to scale the desired width.

We specify the desired confidence interval $[\tilde{p}_{L_k}, \tilde{p}_{U_k}]$ for the proportions p_k by specified the desired half-width δ_y for the y_k's. Instead of a common δ_y, we might specify a separate δ_{y_k} for each y_k, but we believe that in most cases a common δ_y is appropriate. When the desired confidence interval for y_k is $[\hat{y}_k - \delta_y, \hat{y}_k + \delta_y]$, then the desired confidence interval for p_k is $[\tilde{p}_{L_k}, \tilde{p}_{U_k}]$:

$$\tilde{p}_{L_k} = \sin^2(\max(0, \arcsin\sqrt{\hat{p}_k} - \delta_y/2)) \ , \tag{3}$$

and

$$\tilde{p}_{U_k} = \sin^2(\min(\pi/2, \arcsin\sqrt{\hat{p}_k} + \delta_y/2)) \ . \tag{4}$$

The analyst may find it difficult to choose a suitable δ_y. On solution is to specify $\tilde{p}_{L_k}$ or $\tilde{p}_{U_k}$ for a specific $\hat{p}_k$ and to solve δ_y from (3) or (4). For example, if we want that at $\hat{p}_k = 1/2$ $\tilde{p}_{L_k} = 1/2 - \delta_p$, the $\delta_y = \arcsin(2\delta_p)$. Figure 3 demonstrates how the desired confidence interval $[\tilde{p}_{L_k}, \tilde{p}_{U_k}]$ depends on $\hat{p}_k$ when $\delta_y = \arcsin(0.1)$.

In addition to δ_y we must specify the confidence level. We specify a lower bound $1-\alpha$ for the joint coverage of the desired confidence intervals:

$$\Pr\{p_1 \in [\tilde{p}_{L_1}, \tilde{p}_{U_1}] \wedge \ldots \wedge p_m \in [\tilde{p}_{L_m}, \tilde{p}_{U_m}]\} \ge 1 - \alpha \ . \tag{5}$$

The termination rule we propose is very similar to the traditional rule applied when a single mean is estimated. The simulation will be continued until each desired confidence interval $[\tilde{p}_{L_k}, \tilde{p}_{U_k}]$ includes the corresponding estimated confidence interval $[\hat{p}_{L_k}, \hat{p}_{U_k}]$, that is $\tilde{p}_{L_k} \le \hat{p}_{L_k} < \hat{p}_{U_k} \le \tilde{p}_{U_k}$, $k = 1, \ldots, m$, where

$$\begin{aligned} \hat{p}_{L_k} &= \sin^2(\max(0, \hat{y}_{L_k}/2)) \ , \\ \hat{p}_{U_k} &= \sin^2(\min(\pi/2, \hat{y}_{U_k}/2)) \ , \\ \hat{y}_{L_k} &= 2\arcsin\sqrt{\hat{p}_k} \\ &\quad -t_{\nu_k}(1-\alpha_k/2)s_{\hat{p}_k}/\sqrt{\hat{p}_k - \hat{p}_k^2} \ , \\ \hat{y}_{U_k} &= 2\arcsin\sqrt{\hat{p}_k} \\ &\quad +t_{\nu_k}(1-\alpha_k/2)s_{\hat{p}_k}/\sqrt{\hat{p}_k - \hat{p}_k^2} \ , \end{aligned} \tag{6}$$

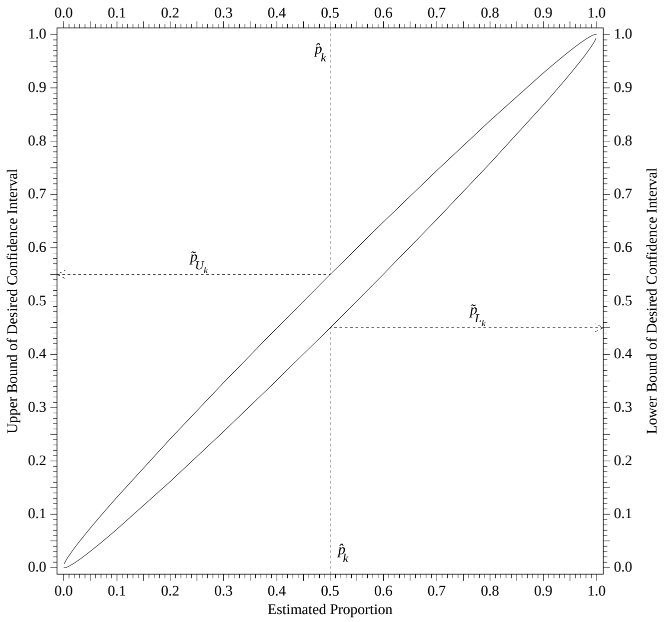


Fig. 3. Desired confidence intervals $[\tilde{p}_{L_k}, \tilde{p}_{U_k}]$ for proportion p_k when the desired confidence interval for $y_k = 2\arcsin\sqrt{p_k}$ is $[\hat{y}_k - \delta_y, \hat{y}_k + \delta_y]$ and $\delta_y = \arcsin(0.1) \approx 00.10017$.

and $s^2_{\hat{p}_k}$ is the estimated variance of $\hat{p}_k$ with ν_k degrees of freedom. When the α_k's in (6) are arbitrary positive constants such that $\sum_{k=1}^m \alpha_k = \alpha$, then we can conclude that the requirement in 5 is satisfied. The conclusion is based on the Bonferroni inequality; see e.g. [12, Ch. 4.8] or [4, Ch. 9.7].

The most common way of applying the Bonferroni inequality is to choose $\alpha_k = \alpha/m$, $k = 1, \ldots, m$. However, this choice is usually inefficient, since typically only a few proportions dominate the run length. Therefore, we propose an alternative way of applying the Bonferroni inequality. Instead of choosing $\alpha_k = \alpha/m$, we estimate the confidence levels of the desired confidence intervals $[\tilde{p}_{L_k}, \tilde{p}_{U_k}]$ or $\hat{\alpha}_k = 1 - \widehat{\Pr}\{p_k \in [\tilde{p}_{L_k}, \tilde{p}_{U_k}]\}$. When we use the Student t-distribution to approximate the distribution of $(\hat{y}_k - y_k)/s_{\hat{y}_k}$, we obtain

$$\hat{\alpha}_k = 2F_{\nu_k}\left(-\delta_y\sqrt{\hat{p}_k(1-\hat{p}_k)}\Big/ s_{\hat{p}_k}\right) , \tag{7}$$

where F_{ν_k} is the cumulative distribution function of the Student t-distribution with ν_k degrees of freedom. The termination rule we propose is that the simulation is continued until

$$\sum_{k=1}^{m} \hat{\alpha}_k \leq \alpha , \tag{8}$$

where the $\hat{\alpha}_k$'s are computed through (7).

D. Small Proportions

The termination rule (8) can not be used if $\Pr\{X \in \mathcal{C}_k\}$ is very close to 0 or 1. In such a case it will be highly probable that $\hat{p}_k(1-\hat{p}_k) = 0$, which implies that Eq. (7) provides $\hat{\alpha}_k = 2F_{\nu_k}(0) = 1$. Therefore, the simulation should be continued forever. Moreover, the sequence $\{0, \ldots, 0\}$ or $\{1, \ldots, 1\}$ does not provide any information about $\mathrm{Var}(\hat{p}_k)$.

When we have observed the sequence $I_{k,j} = 0$, $j = 1, \ldots, n$, and we assume that the $I_{k,j}$'s are independent, the confidence interval for p can be based on the fact that $\Pr\{I_{k,1} = 0 \wedge \cdots \wedge I_{k,n} = 0\} = (1-p_k)^n$. The $100(1-\alpha_k)$ percent confidence interval for p_k is $[0, \hat{p}_{U_k}]$, where $\hat{p}_{U_k} = 1 - \sqrt[n]{\alpha_k}$; see e.g. [13].

Since in steady-state simulation it is very dangerous to assume that the observations are independent, we should learn the dependencies between the $I_{k,j}$'s. However, the sequence $\{0, \ldots, 0\}$ does not contain any information about the dependencies. Therefore, we propose the following practical treatment of small proportions.

We propose that the analyst specifies a tolerance of small proportion p_{small}. If $\hat{p}_k < p_{small}$ or $\hat{p}_k > 1 - p_{small}$, then we do not use (7) but we set

$$\hat{\alpha}_k = \begin{cases} 0, & \text{if } np_{small} \geq 16 , \\ 1, & \text{if } np_{small} < 16 . \end{cases}$$

By setting $\hat{\alpha}_k = 1$ we force the simulation be continued so that reliable estimates for all $p_k \in [p_{small}, 1 - p_{small}]$ may be obtained. The constant 16 is based the results presented in [7].

IV. Example Applications

As examples of estimating the dynamic properties in queueing systems, we simulated three different models: an M/M/1-queue, an M/D/1-queue, and an imbedded autoregressive process.

In the M/M/1-queue with arrival rate $\lambda = 1$ and service rate $\mu = 1.25$ we examined the queue length. The threshold level was 0.5 which implies that we are interested in the durations of idle and busy periods. The sampling interval was 1.

In the M/D/1-queue with $\lambda = 125$ and a constant service time $S = 0.002$ we examined the server utilization. The threshold level was 0.25 which is the steady-state utilization. The sampling interval was 1.

The imbedded AR-process examined models the packet rate to and from a file server. The model is based on measurement data. The derivation of the imbedded AR-process is described in the Appendix. The threshold level was 100 pps which is quite close to the steady-state mean.

With each model we executed 101 independent simulation runs. The parameters controlling the precision were $\alpha = 0.1$, $\delta_y = \arcsin(0.1)$ implying that at $\hat{p}_k = 0.5$ the desired confidence interval is $[0.45, 0.55]$. The tolerance of small probability was $p_{small} = 0.01$.

One practical problem in estimating a histogram is the specification of class boundaries. We use a $\log_2$ related classification (called *octave bins* in [8, p. 552]), in which

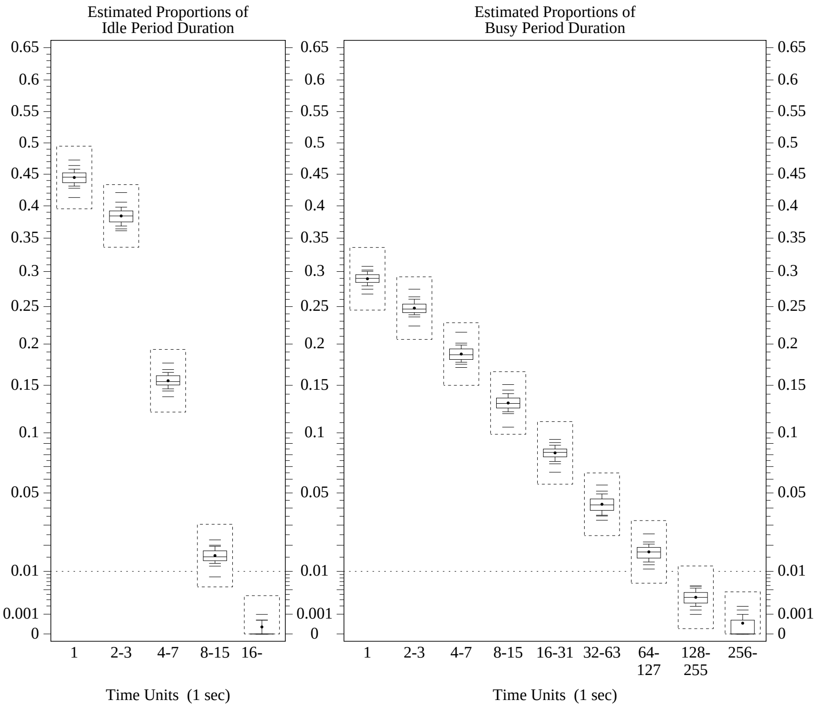


Fig. 4. Boxplots of estimated proportions for histograms of the Idle and Busy Period Durations in M/M/1-queue with $\lambda = 1$ and $\mu = 1.25$
Target Precision: 90% of proportions simultaneously inside the dashed boxes; $\alpha = 0.1$, $\delta_y = \arcsin(0.1)$, $p_{small} = 0.01$

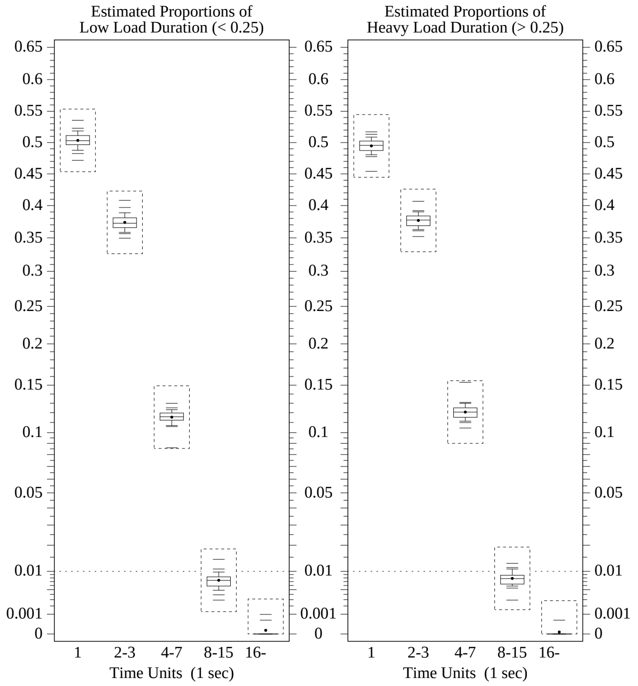


Fig. 5. Boxplots of estimated proportions for histograms of the Low and Heavy Load Durations in M/D/1-queue with $\lambda = 125$ and $S = 0.002$
Target Precision: 90% of proportions simultaneously inside the dashed boxes; $\alpha = 0.1$, $\delta_y = \arcsin(0.1)$, $p_{small} = 0.01$

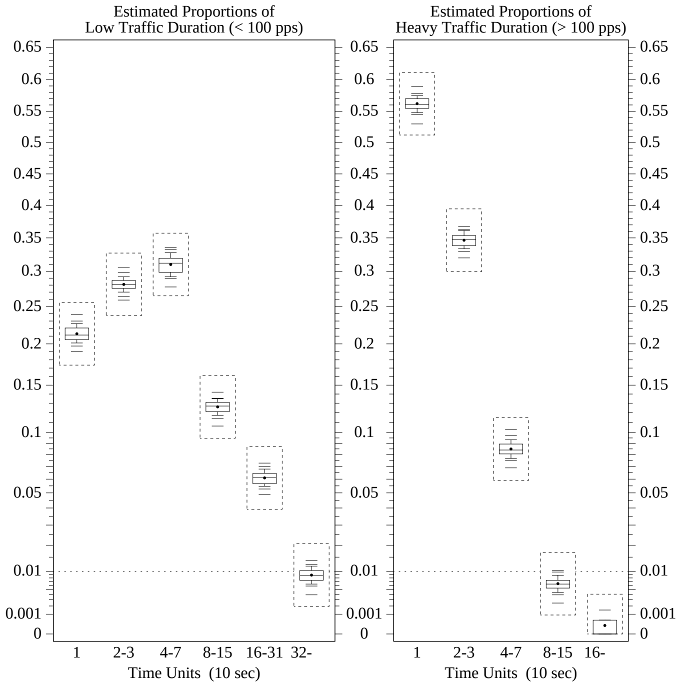


Fig. 6. Boxplots of estimated proportions for histograms of the Low and Heavy Traffic Durations in the Imbedded AR-Model
Target Precision: 90% of proportions simultaneously inside the dashed boxes; $\alpha = 0.1$, $\delta_y = \arcsin(0.1)$, $p_{small} = 0.01$

Table 1. Frequencies of the Number of Cycles

Number of Cycles	**M/M/1 -queue**	**M/D/1 -queue**	**Imbedded AR-model**
2,048*	97	101	95
4,096	4		6

*: Minimum number of cycles when $p_{small} = 0.01$ and at least one estimated proportion is less than p_{small}

Table 2. Comparison of run lengths when proportions and means are estimated

proportions estimated: $\alpha = 0.1$, $\delta_y = \arcsin(0.1)$
mean estimated: $\alpha = 0.1$, $\varepsilon = 0.1$

	M/M/1 -queue		M/D/1 -queue		Imbedded AR-model	
Run Length	Proportion	Mean	Proportion	Mean	Proportion	Mean
Mean	23,978	46,070	8,194	256	15,925	1,212
St.dev.	4,544	29,137	77	0	3,536	637
Min	20,801	16,384	7,963	256	14,480	256
Max	46,809	131,072	8,360	256	30,519	4,096

the boundaries are powers of 2. This classification is usually a good ad hoc guess if the analyst believes that the distribution may be (very) long-tailed.

We also estimated the corresponding means: the mean queue length in the M/M/1 queue, the mean utilization of the server in the M/D/1 queue, and the mean packet rate in the imbedded AR-model. The objective of estimating also the means was to learn how hard a problem the simultaneous estimation of several proportions is when compared to the estimation of a single mean. When the means were estimated, the parameters controlling the precision were $\alpha = 0.1$ and $\varepsilon = 0.1$.

The estimated proportions for the histograms are summarized in Figures 4–6 which provide the boxplots of estimated proportions. In the boxplot a bullet '$\bullet$' denotes the mean of estimates. The three quartiles are visualized by a box ' '. The whiskers ' ' reach from the 25$^{\text{th}}$ to the 10$^{\text{th}}$ percentile and from the 75$^{\text{th}}$ to the 90$^{\text{th}}$ percentile. The smallest and largest estimate together with the 5$^{\text{th}}$ and 95$^{\text{th}}$ percentile are marked by ticks '$|$'. The dashed boxes denotes the values of $\hat{p}_k$, for which $p_k \in [\tilde{p}_{L_k}, \tilde{p}_{U_k}]$.

As the figures indicate all the estimated proportions satisfy the target precision. This is essentially more than requested but not surprising since the Bonferroni inequality provides an lower bound for the confidence level. The second factor that has increased the observed coverage is the treatment of small proportions. With $p_{small} = 0.01$ the minimum number of cycles is 2,048 if at least one estimated proportion is less than 0.01. In each model we have also estimated proportions less than 0.01.

The frequencies of the number of cycles, that is the lengths of indicator sequences used in estimating the variances of estimated proportions, are given in Table 1. As the table indicates the number of cycles is quite stable. In these examples the treatment of small proportions has dominated the lengths of simulation runs.

In order to asses the amount of computing required in estimating several proportions, we also simulated the corresponding means. The run lengths are compared in Table 2. The table indicates that the estimation of several proportions can be considerably cheaper or more expensive than the estimation the corresponding mean when $\alpha = 0.1$, $\varepsilon = 0.1$, and $\delta_y = \arcsin(0.1)$. It should be noted that the $\varepsilon = 0.1$ and $\delta_y = \arcsin(0.1)$ are not necessarily equivalent. However, the comparison shows that the estimation of two histograms does not necessarily require simulation runs of great lengths.

If we compare the computer times needed to estimate several proportions and a single mean, the proportions are more expensive. This is due to the variance estimation. For each proportion we need an estimated variance. However, the spectral method can be implemented so that multiple variance estimates can be calculated efficiently.

Summary

We have proposed a way of characterizing and estimating the dynamic properties of a queueing system. The method is based on translating the problem into simultaneous estimation of several proportions for two histograms. This allows us to apply the Bonferroni inequality in an efficient way and to use existing methods for estimating the variances.

The example applications demonstrate that the method controls the run length so that the estimated proportions have the precision requested. We have not presented theoretical analysis since it can be found in [7].

Appendix: Imbedded Autoregressive Process

The imbedded autoregressive process used as the example in Section IV was constructed in the following way.

1) Identification of the imbedded autoregressive process consists of three phases.

In the first phase the measured sequence $\{x_j\}_{j=1}^n$ is replaced by the normal scores $\{z_j\}_{j=1}^n$: If the k^{th} order statistic is x_j, i.e. $x_j = x_{(k)}$, then $z_j = \Phi^{-1}(\frac{k}{n+1})$, where $\Phi(z)$ is the standard normal distribution function.

In the second phase, the order p of the autoregressive model, $Z_t = \sum_{k=1}^{p} \beta_k Z_{t-k} + \beta_0 \epsilon_t$, is determined using Akaike's final prediction error (FPE) criterion [14].

In the third phase the NAG Library routine G13ACF [15] is used to estimate the parameters $\beta_0, \ldots, \beta_p$. Figure 7 gives the parameters and shows the FPE-values.

2) Generation of the normal autoregressive process $\{Z_t\}$'s is done using the NAG Library routines G05EGF (setup routine) and G05EWF (generation routine). The imbedded AR-process $\{X_t\}$ was constructed using the transform:

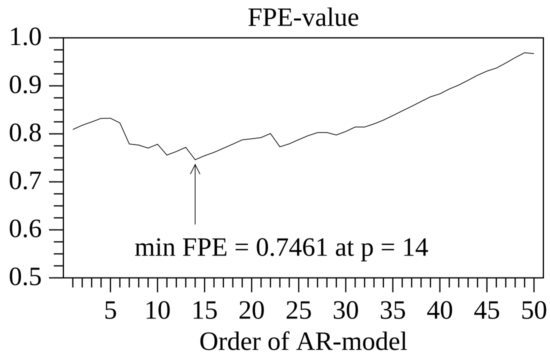
$$X_t = F_X^{-1}[\Phi(Z_t)] \,,$$

where

$$F_X^{-1}(p) = x_{(j-1)}[j - p(n+1)] + x_{(j)}[p(n+1) - j + 1] \,;$$

when

$$\frac{j-1}{n+1} \le p \le \frac{j}{n+1} \,, \; j = 1, \ldots, n+1 \,,$$



Parameters of the imbedded AR process

k	β_k	k	β_k	k	β_k
0	.6373	5	.0757	10	.0396
1	.4060	6	.0603	11	-.2236
2	-.0416	7	.2318	12	.0465
3	.0847	8	-.0641	13	-.0860
4	.0570	9	-.1398	14	.2103

Fig. 7. Summary of the Identified Imbedded Autoregressive Process

and $x_{(0)}$ and $x_{(n+1)}$ are interpolated:

$$\begin{aligned} x_{(0)} &= 2x_{(1)} - x_{(2)}\,, \\ x_{(n+1)} &= 2x_{(n)} - x_{(n-1)}\,. \end{aligned}$$

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